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On the stability of the critical state with inhomogeneous temperature in composite superconductors

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Abstract. The problem of the thermal and magnetic destruction of the critical state in composite superconductors is investigated. The initial distributions of temperature and electromagnetic field are assumed to be essentially inhomogeneous. The limit of the thermomagnetic instability in quasi-stationary approximation is determined. The obtained integral criterion, unlike the analogous criterion for a homogeneous temperature profile, is shown to take into account the influence of any part of the superconductor on the threshold for critical-state instability.

PACS. 74.60.Ge Flux pinning, flux creep, and flux-line lattice dynamics – 74.20.De Phenomenological theories (two-fluid, Ginzburg-Landau, etc.) – 74.25.Ha Magnetic properties

While dealing with instabilities of the critical state in hard superconductors, the character of the temperature distribution $T(x,t)$ and that of the electromagnetic field $\boldsymbol{E}(x,t)$ are of substantial practical interest [1]. This derives from the fact that thermal and magnetic distractions of the critical state caused by Joule self-heating are defined by the initial temperature and electromagnetic field distributions. Hence, the form of the temperature profile may noticeably influence the criteria of critical-state stability with respect to jumps in the magnetic flux in a superconductor. Earlier (*cf.*, *e.g.*, [2]), in dealing with this problem, it was usually assumed that the spatial distribution of temperature and field were either homogeneous or slightly inhomogeneous. However, in reality, physical parameters of superconductors may be inhomogeneous along the sample as well as in its cross-sectional plane. Such inhomogeneities can appear due to different physical reasons. First, the vortex structure pinning can be inhomogeneous due to the existence of weak bonds in the superconductor. Second, inhomogeneity of the properties may be caused by their dependence on the magnetic field H. Indeed, the field H influences many physical quantities, such as the critical current density j_c, the differential conductivity σ_d, and the heat conductivity k.

In the present paper, the temperature distribution in the critical state of composite superconductors is investigated in the quasi-stationary approximation. It is shown that the temperature profile can be essentially inhomogeneous, which affects the conditions of initiation of magnetic flux jumps.

The evolution of thermal (T) and electromagnetic ($\boldsymbol{E}, \boldsymbol{H}$) perturbations in superconductors is described by a nonlinear heat conduction equation [3],

$$\nu \frac{\mathrm{d}T}{\mathrm{d}t} = \nabla[\kappa \nabla T] + \boldsymbol{j}\boldsymbol{E}, \tag{1}$$

a system of Maxwell's equations,

$$\mathrm{rot}\boldsymbol{E} = -\frac{1}{c}\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t}, \tag{2}$$

$$\mathrm{rot}\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} \tag{3}$$

and a critical-state equation

$$\boldsymbol{j} = \boldsymbol{j}_c(T, \boldsymbol{H}) + \boldsymbol{j}_r(\boldsymbol{E}). \tag{4}$$

Here $\nu = \nu(T)$ is the specific heat, $\kappa = \kappa(T)$ is the thermal conductivity respectively; $\boldsymbol{j}_c$ is the critical current density and $\boldsymbol{j}_r$ is the active current density.

We use the Bean-London critical state model to describe the $j_c(T,H)$ dependence, according to which $j_c(T) = j_0 - a(T - T_0)$ [4], where the parameter a characterizes thermally activated weakening of Abrikosov vortex pinning on crystal lattice defects, j_0 is the equilibrium current density, and T_0 is the temperature of the superconductor.

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The dependence $j_r(E)$ in the region of sufficiently strong electric fields ($E \geq E_f$; where E_f is the limit of the linear region of the current-voltage characteristic of the sample [2]) can be approximated by a piecewise-linear function $j_r \approx \sigma_f E$, where $\sigma_f = \dfrac{\eta c^2}{H\Phi_0} \approx \dfrac{\sigma_n H_{c_2}}{H}$ is the effective conductivity in the flux flow regime and η is the viscous coefficient, $\Phi_0 = \dfrac{\pi hc}{2e}$ is the magnetic flux quantum, σ_n is the conductivity in the normal state, H_{c_2} is the upper critical magnetic field. In the region of the weak fields ($E \leq E_f$), the function $j_r(E)$ is nonlinear. This nonlinearity is associated with thermally activated creep of the magnetic flux [5].

Let us consider a superconducting sample placed into an external magnetic field $\boldsymbol{H} = (0, 0, H_e)$ increasing at a constant rate $\dfrac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = \dot{H} = \text{const}$. According to the Maxwell equation (2), a vortex electric field $\boldsymbol{E} = (0, E_e, 0)$ is present. Here H_e is the magnitude of the external magnetic field and E_e is the magnitude of the back-ground electric field. In accordance with the concept of the critical state, the current density and the electric field must be parallel: $\boldsymbol{E} \parallel \boldsymbol{j}$; where H_e is the amplitude of the external magnetic field and E_e is the amplitude of the external electric field.

The thermal and electromagnetic boundary conditions for equations (1–4) have the form

$$\kappa \frac{\mathrm{d}T}{\mathrm{d}x}\bigg|_{x=0} + w_0[T(0) - T_0] = 0, \qquad T(L) = T_0, \tag{5}$$

$$\frac{\mathrm{d}E}{\mathrm{d}x}\bigg|_{x=0} = 0, \qquad E(L) = 0.$$

For the plane geometry (Fig. 1) and for the boundary conditions $H(0) = H_e$, $H(L) = 0$, the magnetic field distribution is $H(x) = H_e(L - x)$, where $L = \dfrac{cH_e}{4\pi j_c}$ is the depth of magnetic flux penetration into the sample and w_0 is the coefficient of heat transfer to the cooler at the equilibrium temperature T_0.

The condition of applicability of equations (1–4) to the description of the dynamics of evolution of thermomagnetic perturbations are discussed at length in [1].

In the quasi-stationary approximation, terms with time derivatives can be neglected in equations (1–4). This means that the heat transfer from the sample surface compensates the energy dissipation arising in the viscous flow of magnetic flux in the medium with an effective conductivity σ_f. In this approximation, the solution to equation (2) has the form

$$E = \frac{\dot{H}}{c}(L - x). \tag{6}$$

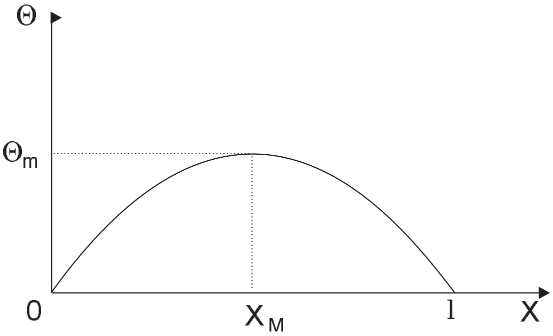

Fig. 1. The distribution of the temperature profile $\Theta(x)$.

Upon substituting this expression into equation (1) we get an inhomogeneous equation for the temperature distribution $T(x,t)$,

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\rho^2} - \rho\Theta = f(\rho). \tag{7}$$

Here we introduced the following dimensionless variables

$$f(\rho) = -[1 + r\omega\rho]\frac{j_0}{aT_0}, \quad \Theta = \frac{T - T_0}{T_0}, \quad \rho = \frac{L - x}{r},$$

and the dimensionless parameters $\omega = \dfrac{\sigma_f \dot{H}}{cj_0}$, and, $r = \left(\dfrac{c\kappa}{a\dot{H}L^2}\right)^{1/3}$, where r characterizes the spatial scale of the temperature profile inhomogeneity in the sample. Solutions to equation (7) are Airy functions, which can be expressed through Bessel functions of the order 1/3 [6]

$$\Theta(\rho) = C_1\rho^{1/2}K_{1/3}\left(\frac{2}{3}\rho^{3/2}\right) + C_2\rho^{1/2}I_{1/3}\left(\frac{2}{3}\rho^{3/2}\right) + \Theta_0(\rho), \tag{8}$$

$$\Theta_0(\rho) = \rho^{1/2}K_{1/3}\left(\frac{2}{3}\rho^{3/2}\right) \times \int_0^\rho [1 + r\omega\rho_1]\rho_1^{3/2} I_{1/3}\left(\frac{2}{3}\rho_1^{3/2}\right)\mathrm{d}\rho_1 - \rho^{1/2}I_{1/3}\left(\frac{2}{3}\rho^{3/2}\right)\int_0^\rho [1 + r\omega\rho_1]\rho_1^{3/2}K_{1/3}\left(\frac{2}{3}\rho_1^{3/2}\right)\mathrm{d}\rho_1,$$

where C_1 and C_2 are integration constants, which are determined by the boundary conditions to be

$$C_1 = 0, \quad C_2 = \frac{-w_0 L\Theta(0) + \kappa \dfrac{\mathrm{d}\Theta}{\mathrm{d}\rho}\bigg|_{\rho=\frac{L}{r}}}{\left[w_0\left(\dfrac{L}{r}\right)^{1/2} I_{1/3}\left(\dfrac{2}{3}\rho^{-3/2}\right) - 2\dfrac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^{1/2}I_{1/3}\left(\dfrac{2}{3}\rho^{3/2}\right)\right)\right]\bigg|_{\rho=\frac{L}{r}}}.$$

From the Maxwell equation (2), the temperature inhomogeneity parameter can be expressed in the form

$$\alpha = \frac{r}{L} = \left[\frac{4\pi\nu j_0}{aH_e^2}\frac{H_e}{\dot{H}t_\kappa}\right]^{1/3}. \tag{9}$$

It is evident that $\alpha \sim 1$ near the threshold for a flux jump, when $\frac{aH_e^2}{4\pi\nu j_0} \sim 1$, even under the quasi-stationary heating condition $\frac{\dot{H}t_\kappa}{H_e} \ll 1$; where $t_\kappa = \frac{\nu L^2}{\kappa}$ is the characteristic time of the heat conduction problem.

Let us estimate the maximum heating temperature Θ_m in the isothermal case $w = \frac{\kappa}{L} \geq 1$. The solution to equation (7) can be represented in the form

$$\Theta(x) = \Theta_m - \rho_0\frac{(x-x_m)^2}{2}, \tag{10}$$

near the point at which the temperature is a maximum, $x = x_m$ (Fig. 1).

With solution (10) being approximated near the point $x_m = \frac{L}{2}$ with the help of the thermal boundary conditions, the coefficient ρ_0 can be easily determined to be $\left(\frac{8}{L^2}\right)\Theta_m$ and the temperature can be written as

$$\Theta(x) = \Theta_m\left[1 - \frac{4}{L^2}\left(x - \frac{L}{2}\right)^2\right]. \tag{11}$$

Substituting this solution into equation (7), the superconductor maximum heating temperature due to magnetic flux jumps can be estimated as

$$\Theta_m = \frac{\left[j_0 + \frac{\sigma_f\dot{H}}{c}(L-x_m)\right]\frac{\dot{H}}{c\kappa T_0}(L-x_m)}{\frac{\gamma}{L^2} + \frac{a\dot{H}}{c\kappa}(L-x_m)}. \tag{12}$$

For a typical situation, when $\frac{\gamma}{L^2} \ll \frac{a\dot{H}}{c\kappa}(L-x_m)$ the estimation for Θ_m is

$$\Theta_m \approx \left[j_0 + \frac{\sigma_f\dot{H}}{c}(L-x_m)\right]\frac{\dot{H}L^2}{c\kappa T_0}(L-x_m). \tag{13}$$

Here, the parameter $\gamma \sim 1$ (for a parabolic temperature profile $\gamma \sim 8$). It is easy to verify that for typical values of $j_0 = 10^6$ A/cm^2, $\dot{H} = 10^4$ G/s, and $L = 0,01$ cm the heating is sufficiently low: $\Theta_m \ll 1$. In the case of poor sample cooling, $w = 1 \div 10$ erg/(cm^2 s K), the Θ_m is

$$\Theta_m = \frac{\dot{H}j_0L^2}{cw_0T_0} \approx 0,5;$$

i.e., the heating temperature can be as high as $\delta T_m = T_0\Theta_m \sim 2K$. One can see that in the case of poor sample cooling, the heating can be rather noticeable and influences the conditions of the thermomagnetic instability of the critical state in the superconductor.

Let us investigate the stability of the critical state with respect to small thermal δT and electromagnetic δE fluctuations in the quasi-stationary approximation. We represent solutions to equations (1–4) in the form

$$T(x,t) = T(x) + \exp\left\{\frac{\lambda t}{t_\kappa}\right\}\delta T\left(\frac{x}{L}\right),$$
$$E(x,t) = E(x) + \exp\left\{\frac{\lambda t}{t_\kappa}\right\}\delta E\left(\frac{x}{L}\right), \tag{14}$$

where $T(x)$ and $E(x)$ are solutions to the unperturbed equations obtained in the quasi-stationary approximation describing the background distributions of temperature and electric field in the sample and λ is a parameter to be determined. The instability region is determined by the condition that Re$\lambda \geq 0$. From solution (14), one can see that the characteristic time of thermal and electromagnetic perturbations t_j is of the order of t_κ/λ. Linearizing equations (1–4) for small perturbations $\left(\frac{\delta T}{T(x)}, \frac{\delta E}{E(x)} \ll 1\right)$ we obtain the following equations in the quasi-stationary approximation:

$$\nu\frac{\lambda}{t_\kappa}\delta T = \frac{\kappa}{L^2}\frac{\mathrm{d}^2\delta T}{\mathrm{d}x^2} + [j(x) + \sigma_f E(x)]\,\delta E - aE(x)\delta T,$$
$$\frac{1}{L^2}\frac{\mathrm{d}^2\delta E}{\mathrm{d}x^2} = \frac{4\pi\lambda}{c^2t_\kappa}[\sigma_f\delta E - a\delta T]. \tag{15}$$

Eliminating the variable δT between equations (15), we obtain a fourth-order differential equation with variable coefficients for the electromagnetic field δE:

$$\frac{\mathrm{d}^4\delta E}{\mathrm{d}z^4} - \left[\lambda(1+\tau) + \frac{E(z)}{E_\kappa}\right]\frac{\mathrm{d}^2\delta E}{\mathrm{d}z^2} + \lambda[\lambda\tau - B(z)]\,\delta E = 0. \tag{16}$$

Here, we introduced the following dimensionless variables:
$z = \frac{x}{L}$, $B(z) = \frac{4\pi aL^2}{c^2\nu}j(z)$, $j(z) = \sigma_f E(z) - [j_0 - a(T(z)-T_0)]$, $E(z) = \frac{\dot{H}L}{c}(1-z)$, $\tau = \frac{4\pi\sigma_f\kappa}{c^2\nu}$, $E_\kappa = \frac{\kappa}{aL^2}$. $\nu = \nu_0\left(\frac{T}{T_0}\right)^3$, $\kappa = \kappa_0\left(\frac{T}{T_0}\right)$.

One should keep in mind that the variable $T(z)$ and $E(z)$ are given by equation (8), in which $\rho = \frac{L}{r}(1-z)$. Using the relation between δT and δE given by equations (15), we write the boundary conditions to equation (16) in the form:

$$\left.\frac{\mathrm{d}^2\delta E}{\mathrm{d}z^2}\right|_{z=1}=0,\quad \left.\frac{\mathrm{d}^3\delta E}{\mathrm{d}z^3}\right|_{z=0}=-W\left.\left[\frac{\mathrm{d}^2\delta E}{\mathrm{d}z^2}-\lambda\tau\delta E\right]\right|_{z=0},$$

$$\delta E|_{z=1}=0,\quad \left.\frac{\mathrm{d}\delta E}{\mathrm{d}z}\right|_{z=0}=0 \tag{17}$$

where $W=\frac{w_0 L}{\kappa}$ is the dimensionless thermal impedance.

The condition for the existence of a nontrivial solution to equation (16) subject to boundary conditions (17) allows one to determine the boundary of the critical-state thermomagnetic instability in a superconducting sample. This problem is complicated, and its analytical solution cannot be found in a closed form. We will consider the development of thermomagnetic instability in the dynamical approximation, which is valid for composite superconductors with high value of $\sigma_f E$.

The dynamical character of the instability development leads to the predominance of heat diffusion over magnetic flux diffusion in the sample: $\tau=\frac{D_t}{D_m}\gg 1$ [1], where $D_t=\frac{\kappa}{\nu}$ and $D_m=\frac{c^2}{4\pi\sigma_f}$ are the coefficients of the thermal and magnetic diffusion, respectively.

In this case, as seen from equation (14), the characteristic times t_j of temperature and electromagnetic field perturbations have to satisfy the inequalities $t_j\gg t_\kappa$ ($\lambda\ll 1$) and $t_j\ll t_m$ ($\lambda\tau\gg 1$), where $t_\kappa=\frac{L^2}{D_t}$ and $t_m=\frac{L^2}{D_m}$ are the characteristic times of the thermal and magnetic diffusion, respectively.

As well known that [1], the effective magnitude of conductivity σ_f is greater in composite superconductors than the one in hard superconductors. We can assume that induced normal current $\sigma_f E$ compensates the decreasing of critical current $j_c(T)$, caused by increasing of temperature and obviously prevents magnetic flux penetration into the sample. In this case we can neglect the moving of magnetic flux. In the other word, thermomagnetic instability develops slowly as compared with the heat diffusion with characteristic time of increasing $t_j\sim\frac{t_\kappa}{\lambda}\gg t_\kappa$ ($\lambda\ll 1$) or $t_m\gg\frac{t_m}{\lambda\tau}=\frac{t_\kappa}{\lambda}=t_j$ ($\lambda\tau\gg 1$).

In the approximation ($\tau\gg 1$, $\lambda\tau\gg 1$, $\lambda\ll 1$) equation (16) is reduced to a lower order differential equation

$$\frac{\mathrm{d}^2\delta E}{\mathrm{d}z^2}+\left[\lambda-\frac{B(z)}{\tau}\right]\delta E=0. \tag{18}$$

In the case of $\tau\gg 1$, the instability threshold depends on the electrodynamic boundary conditions at the surface of the sample only slightly. Therefore, the electrodynamic boundary conditions at the boundaries of the current-carrying layer ($z=0, z=1$) can be neglected and one can keep only the thermal boundary conditions to equation (18).

Multiplying equation (18) by δE and integrating the result with respect to z over the interval $0<z<1$, we obtain

$$\lambda=\frac{\frac{1}{\tau}\int_0^1 B(z)\delta E^2\mathrm{d}z-\int_0^1\left(\frac{\mathrm{d}\delta E}{\mathrm{d}z}\right)^2\mathrm{d}z}{\int_0^1\delta E^2\mathrm{d}z} \tag{19}$$

where we use the equality

$$\int_0^1\frac{\mathrm{d}^2\delta E}{\mathrm{d}z^2}\delta E\mathrm{d}z=\delta E(z)\left(\frac{\mathrm{d}\delta E}{\mathrm{d}z}\right)-\int_0^1\left(\frac{\delta E}{\mathrm{d}z}\right)^2\mathrm{d}z$$
$$=-\int_0^1\left(\frac{\delta E}{\mathrm{d}z}\right)^2\mathrm{d}z$$

and the boundary conditions. The right-hand side of equation (19) has a minimum at $\lambda=\lambda_c$:

$$\lambda_c=\frac{\frac{1}{\tau}\int_0^1 B(z)\delta E^2\mathrm{d}z}{\int_0^1\delta E^2\mathrm{d}z}, \tag{20}$$

then

$$\frac{1}{\tau}\int_0^1 B(z)n_E^2\mathrm{d}z=\frac{\int_0^1\left(\frac{\mathrm{d}\delta E}{\mathrm{d}z}\right)^2\mathrm{d}z}{\int_0^1\delta E^2\mathrm{d}z}\cdot \tag{21}$$

Here we introduced the following unit vector

$$n_E^2=\frac{\delta E^2}{\int_0^1\delta E^2\mathrm{d}z},\qquad \int_0^1 n_E^2\mathrm{d}z=1.$$

Since we do not know the function $\delta E(z)$, we try, following [7], to obtain an integral estimation of the instability growth increment and the low boundary of its occurrence. The behavior of the integrand in equation (21) is basically determined by the factor $E=\frac{\dot{H}L}{c}(1-z)$, which is equal to zero at $z=1$ (the other factors change more smoothly). Hence, the integrand

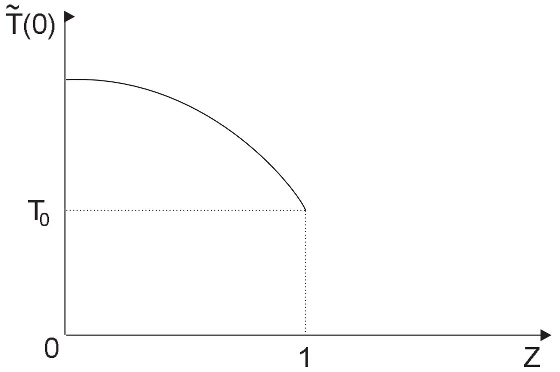

Fig. 2. Plots of the function $T(z)$.

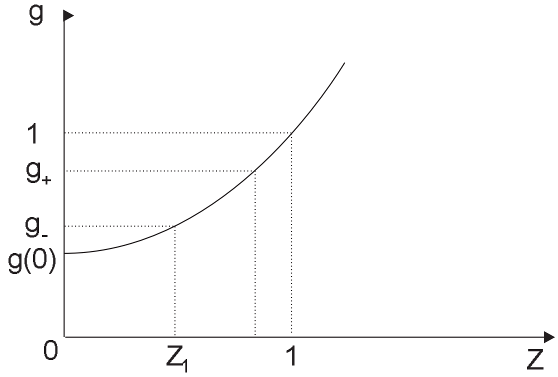

Fig. 3. Plots of the function $g(z)$.

reaches its maximum at $z = 0$ and the upper estimate for λ_c is

$$\lambda_c \leq \frac{B(0)}{\tau} \cdot \tag{22}$$

It is evident that $\lambda_c \ll 1$ and $\lambda_c\tau \gg 1$ at $\tau \gg 1$. Numerical evaluation gives $\lambda_c \approx 0.1 \div 10^{-2}$ at $\tau = 10^3$.

Equations (21, 22) enable one to write the instability occurrence criterion in the form

$$\frac{1}{\tau}\int_0^1 B(z)n_E^2 \mathrm{d}z > \frac{\int_0^1 \left(\frac{\mathrm{d}\delta E}{\mathrm{d}z}\right)^2}{\int_0^1 \delta E^2 \mathrm{d}z} \cdot \tag{23}$$

This criterion essentially depends on the boundary conditions and the functions $j(z)$, $E(z)$, $T(z)$. Figure 2 presents graph of the function $T(z)$. Inequality (23) can be strengthened by means of an evaluation,

$$\int_0^1 \left(\frac{\mathrm{d}\delta E}{\mathrm{d}z}\right)^2 \mathrm{d}z > \frac{\pi^2}{4}\int_0^1 \delta E^2 \mathrm{d}z \tag{24}$$

which can be easily verified by expanding the function $\delta E(z)$ in a Fourier series:

$$\delta E(z) = A_m \cos\frac{\pi z(2m+1)}{2} \cdot$$

Let us now try to strengthen inequality (23) further. For this purpose, we consider the integral

$$\int_0^1 g(z)(n_E^2 - 1)\mathrm{d}z = \int_0^{z_1} g(z)(n_E^2 - 1)\mathrm{d}z + \int_{z_1}^0 g(z)(n_E^2 - 1)\mathrm{d}z.$$

The last term can be represented in the form

$$\int_0^1 g(z)(n_E^2 - 1)\mathrm{d}z = (g_+ - g_-)\int_0^1 g(z)(n_E^2 - 1)\mathrm{d}z,$$

taking intermediate values of the g_- in the range $z < z_1$ and $g > g_-$ in the range $z_1 < z < 1$ outside the integral. It is evident (Fig. 3) that

$$\int_0^1 g(z)(n_E^2 - 1)\mathrm{d}z \leq 0$$

or

$$\int_0^1 g(z)n_E^2 \mathrm{d}z \leq \int_0^1 g(z)\mathrm{d}z. \tag{25}$$

With inequality (25), the instability occurrence criterion can be represented in the form

$$\int_0^1 B(z)\mathrm{d}z \geq \frac{\pi^2}{4}\tau. \tag{26}$$

Inequality (26), unlike the analogous criterion for a homogeneous temperature profile, has an integral character and takes into account the influence of each part of the superconductor on the threshold for the superconducting-state instability. If condition (26) is satisfied, then small fluctuations of temperature δT and electric field δE in the superconductor will exponentially increase with time. The most probable result of the development of such an instability would be a transition from a critical state to a resistive one.

We should emphasis that this result was obtained for an arbitrary temperature dependence of thermophysical parameters ν and κ of superconducting material and for an arbitrary function $j(H)$. Moreover, since the system of equations (1–4) is invariant with respect to an arbitrary translation, the wave propagation condition can be found for an arbitrary critical current density dependence on T and H.

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